

The Instantonic Approach With Non-Equivalent Vacua *

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Abstract

We study the quantum-mechanical tunneling phenomenon in models which include the existence of non-equivalent vacua. For such a purpose we evaluate the euclidean propagator between two minima of the potential at issue in terms of the quadratic fluctuations over the corresponding instantons. The effect of the multi-instanton configurations are included by means of the alternate dilute-gas approximation.

I. INTRODUCTION.

The tunneling through classically forbidden regions represents one of the most striking phenomenon in quantum theory and therefore plays a central role in many areas of modern physics. On the other hand, together with the operator formalism of quantum mechanics we have an equivalent description by means of path-integrals. In such a case the Schrodinger's equation is substituted by a global approach where the quantum mechanical time evolution is analysed in terms of a functional integration. Qualitatively speaking, the path-integral representation corresponds to a sum over all histories allowed to the physical system we are dealing with. To be precise, we need to take into account an imaginary exponential of the classical action associated with every path which fulfill the appropriate initial and end

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points conditions. Of course, the quantum amplitude so built is difficult to handle due to the oscillating character of the exponential at issue. To avoid problems of this sort we carry out the change $t \rightarrow -i\tau$ (known in the literature as the Wick rotation). In doing so, we can take advantage of the euclidean version of the path-integral which represents by itself a new tool for describing relevant aspects of the quantum theory.

Almost from the very beginning of the subject a semiclassical treatment of the tunneling phenomena (ranging from periodic-potentials in quantum mechanics to Yang-Mills models in field theories) has been performed by means of the so-called instantons [1]. Going to more physical terms, the instantons represent localised finite-action solutions of the euclidean equation of motion. To be specific, the euclidean equation is the same as the usual one for our particle in real time except that the potential is turned upside down. Although more massive than the perturbative excitations, the instantons themselves become stable since an infinite barrier separates them from the ordinary sector of the model. The stability is reinforced by the existence of a topological conserved charge which does not arise by Noether's theorem in terms of a well-behaved symmetry, but characterizes the global behaviour of the system when the imaginary time is large enough. Accordingly, it comes by no surprise that these classical solutions have been considered in the literature under the name of topological configurations. Once the appropriate classical solution is well-known, we make an expansion around the topological background to evaluate the quadratic fluctuations which arise in terms of the functional determinant of a second order differential operator. The integration is solved within the gaussian scheme except for the zero-modes which appear due to the invariances of the model. To deal with these excitations we introduce collective coordinates so that ultimately the gaussian integration is carried out only along the directions orthogonal to the zero-modes. As a functional determinant includes an infinite product of eigenvalues, the result should be in principle a highly divergent expression. Fortunately we can regularize the fluctuation factors by means of the ratio of determinants.

To start from scratch let us describe the instanton calculus for the one-dimensional particle as can be found in [2]. The reader finds there a comprehensive description of this

subject. To be specific we assume that our particle moves under the action of a confining potential $V(x)$ which yields a pure discrete spectrum of energy eigenvalues. If the particle is located at the initial time $t_i = -T/2$ at the point x_i while one finds it when $t_f = T/2$ at the point x_f , the functional version of the non-relativistic quantum mechanics allows us to express the transition amplitude in terms of a sum over all paths joining the world points with coordinates $(-T/2, x_i)$ and $(T/2, x_f)$. Making the change $t \rightarrow -i\tau$, known in the literature as the Wick rotation, the euclidean formulation of the path-integral reads

$$\langle x_f | \exp(-HT) | x_i \rangle = N(T) \int [dx] \exp\{-S_e[x(\tau)]\} \quad (1)$$

where H represents the hamiltonian of the model, the factor $N(T)$ serves to normalize the amplitude while the symbol $[dx]$ indicates the integration over all functions which fulfil the boundary conditions at issue. Now we resort to the euclidean action S_e , i.e.

$$S_e = \int_{-T/2}^{T/2} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau \quad (2)$$

whenever the mass of the particle is set equal to unity for notational simplicity. In this approximation we take for granted the existence of a $x_c(\tau)$ configuration which represents a stationary point of the euclidean action. Next we perform the expansion of a generic $x(\tau)$ according to

$$x(\tau) = x_c(\tau) + \sum_j c_j x_j(\tau) \quad (3)$$

where as usual $x_j(\tau)$ stand for a complete set of orthonormal functions

$$\int_{-T/2}^{T/2} x_j(\tau) x_k(\tau) d\tau = \delta_{jk} \quad (4)$$

vanishing at our boundary, i.e. $x_j(\pm T/2) = 0$. The eigenfunctions $x_j(\tau)$ (with eigenvalues ϵ_j) appear associated with the so-called stability equation given by

$$-\frac{d^2 x_j(\tau)}{d\tau^2} + V''[x_c(\tau)] x_j(\tau) = \epsilon_j x_j(\tau) \quad (5)$$

To sum up, the euclidean transition amplitude reduces itself to

$$\langle x_f | \exp(-HT) | x_i \rangle = N(T) \exp(-S_{eo}) \prod_j \epsilon_j^{-1/2} \quad (6)$$

where S_{eo} represents the classical action associated with the configuration $x_c(\tau)$ while the product of eigenvalues is usually written as

$$\prod_j \epsilon_j^{-1/2} = \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + V''[x_c(\tau)] \right] \right\}^{-1/2} \quad (7)$$

according to the conventional notation of the finite-dimensional case. To fix the factor $N(T)$ we resort to a well-known problem where

$$V(x) = \frac{\nu^2}{2} x^2 \quad (8)$$

so that $V''(x=0) = \nu^2$. As corresponds to the harmonic oscillator the relevant amplitude is given by

$$\langle x_f = 0 | \exp(-H_{ho}T) | x_i = 0 \rangle = N(T) \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} \quad (9)$$

Next the evaluation of (9) is possible according to the general method exposed in [3]. In other words

$$\langle x_f = 0 | \exp(-H_{ho}T) | x_i = 0 \rangle = \left(\frac{\nu}{\pi} \right)^{1/2} (2 \sinh \nu T)^{-1/2} \quad (10)$$

II. THE SEXTIC-MODEL.

As anticipated in the introduction the tunneling phenomenon appears as first in the study of systems which have two or more degenerate classical minima separated by potential energy barriers. The standard example would be the point-particle in a double-well potential where, because of tunneling precisely, the two first eigenstates are the spatially even and odd combinations of harmonic oscillators centered at the bottom of the respective wells. In more physical terms, the degeneracy of the energy eigenvalues is broken so that the splitting term results proportional to the barrier-penetration factor. The euclidean configuration which leads the process is of course the instanton whose stability properties are decided by the

behaviour of the small fluctuations in its neighborhood. As a matter of fact the stability is assured by a clever combination of translational invariance and conserved topological charges. In the following we consider the triple-well potential $V(x)$ given by [4]

$$V(x) = \frac{\omega^2}{8} x^2 (x^2 - 1)^2 \quad (11)$$

From a classical point of view we find three minima located at $x_- = -1$, $x_+ = 1$ and $x_o = 0$. Two of them, namely x_- and x_+ , are equivalent since can be connected by means of the discrete symmetry $x \rightarrow -x$ at issue. However the third minimum x_o is invariant under the action of such a symmetry. In other words, the central vacuum is not identical with the others located at both sides. When considering the limit $\omega^2 \gg 1$ the energy barriers are high enough to decompose the system into a sum of independent harmonic oscillators. However the existence of finite barriers between the different wells of the potential yields a relevant tunneling phenomenon so that ultimately the symmetry $x \rightarrow -x$ is not spontaneously broken at quantum level and the expectation value of the coordinate x computed for the ground-state is zero as corresponds to the even character of the potential $V(x)$.

A. The one-instanton contribution.

The question we wish to address at this point should be the explicit description of the tunneling in the euclidean version of the path-integral. As regards the one-instanton amplitude we take into account the transition amplitude between $x_o = 0$ and $x_+ = 1$. For such a purpose we need a classical configuration with $x_i = 0$ at $t_i = -T/2$ while $x_f = 1$ when $t_f = T/2$. It is customarily assumed that $T \rightarrow \infty$ mainly because the explicit solution of the problem is much more complicated for finite T .

First of all we can find the explicit form of the instanton $x_c(\tau)$ just by integration of a first-order differential equation, i.e.

$$\frac{dx_c}{d\tau} = \pm \sqrt{2V(x_c)} \quad (12)$$

where we recognize the quantum mechanical version of the Bogomol'nyi condition [5]. Now we solve (12) by a simple quadrature so that

$$x_c(\tau) = \sqrt{[1 + \tanh \omega(\tau - \tau_c)]/2} \quad (13)$$

where as usual the parameter τ_c indicates the point at which the instanton makes the jump. As expected equivalent solutions are obtained by means of the transformation $\tau \rightarrow -\tau$ and $x_c(\tau) \rightarrow -x_c(\tau)$ so that adjoint minima of the potential can be connected by means of a topological solution in a systematic way. In addition we have that $S_{eo} = \omega/4$. On the other hand we need classical configurations for which $x_- = 0$ and $x_+ = 1$ at large but finite values $\tau = \pm T/2$. However the explicit form of the instantons that appear in the literature corresponds to infinite euclidean time. Fortunately the difference is so small that can be ignored mainly because we are interested in the limit $T \rightarrow \infty$. Our description of the one-instanton amplitude between $x_- = 0$ and $x_+ = 1$ takes over

$$\begin{aligned} \langle x_f = 1 | \exp(-HT) | x_i = 0 \rangle &= N(T) \left\{ \text{Det} \left[-\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} \\ &\left\{ \frac{\text{Det} [-(d^2/d\tau^2) + V''[x_c(\tau)]]}{\text{Det} [-(d^2/d\tau^2) + \nu^2]} \right\}^{-1/2} \exp(-S_{eo}) \end{aligned} \quad (14)$$

where we have multiplied and divided by the determinant associated with the harmonic oscillator of frequency ν . As regards the determinant built over the instanton itself we find a zero-mode $x_o(\tau)$ which could jeopardize the computation procedure as a whole. However this eigenvalue $\epsilon_o = 0$ comes by no surprise since it reflects the translational invariance of the system. We can discover the existence of this zero-mode starting from (13). Including the adequate normalization one can check that

$$x_o(\tau) = \frac{1}{\sqrt{S_{eo}}} \frac{dx_c}{d\tau} \quad (15)$$

is just the solution of (5) with $\epsilon_o = 0$. The way out of this apparent cul-de-sac is simple. The integration over c_o (see (3)) becomes equivalent to the integration over the center of the

instanton τ_c . To fix the jacobian of the transformation at issue we consider a first change like

$$\Delta x(\tau) = x_o(\tau) \Delta c_o \quad (16)$$

Going back to the general expression written in (3) we find that under a shift $\Delta\tau_c$ the global effect corresponds to

$$\Delta x(\tau) = -\sqrt{S_{eo}} x_o(\tau) \Delta\tau_c \quad (17)$$

Finally the identification between (16) and (17) yields

$$dc_o = \sqrt{S_{eo}} d\tau_c \quad (18)$$

where the minus sign disappears since what matters is the modulus of the jacobian at issue.

To sum up we have that

$$\left\{ \frac{Det[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det[-(d^2/d\tau^2) + \nu^2]} \right\}^{-1/2} = \left\{ \frac{Det'[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det[-(d^2/d\tau^2) + \nu^2]} \right\}^{-1/2} \sqrt{\frac{S_{eo}}{2\pi}} d\tau_c \quad (19)$$

where Det' stands for the well known reduced determinant once the zero-mode has been removed. To make an explicit computation of the quotient of determinants we resort to the Gelfand-Yaglom method where only the knowledge of the large- τ behaviour of the classical solution $x_c(\tau)$ is necessary [6]. If \hat{O} and \hat{P} represent a couple of second order differential operators, whose eigenfunctions vanish at the boundary, the quotient of determinants is given in terms of the respective zero-energy solutions $f_o(\tau)$ and $g_o(\tau)$ according to

$$\frac{Det\hat{O}}{Det\hat{P}} = \frac{f_o(T/2)}{g_o(T/2)} \quad (20)$$

whenever the eigenfunctions at issue satisfy the initial conditions

$$f_o(-T/2) = g_o(-T/2) = 0, \quad \frac{df_o}{d\tau}(-T/2) = \frac{dg_o}{d\tau}(-T/2) = 1 \quad (21)$$

For the zero-mode $g_o(\tau)$ associated with the generic harmonic oscillator of frequency ν we have

$$g_o(\tau) = \frac{1}{\nu} \sinh[\nu(\tau + T/2)] \quad (22)$$

so that now we need the explicit form of the solution $f_o(\tau)$ which corresponds to the topological configuration written in (13). Starting from the $x_o(\tau)$ zero-mode we write a second solution $y_o(\tau)$ given by

$$y_o(\tau) = x_o(\tau) \int_0^\tau \frac{ds}{x_o^2(s)} \quad (23)$$

In doing so we may summarize the asymptotic behaviour of $x_o(\tau)$ and $y_o(\tau)$ as follows

$$x_o(\tau) \sim \begin{cases} C \exp(-2\omega\tau) & \text{if } \tau \rightarrow \infty \\ D \exp(\omega\tau) & \text{if } \tau \rightarrow -\infty \end{cases} \quad (24)$$

$$y_o(\tau) \sim \begin{cases} \exp(2\omega\tau)/4\omega C & \text{if } \tau \rightarrow \infty \\ -\exp(-\omega\tau)/2\omega D & \text{if } \tau \rightarrow -\infty \end{cases} \quad (25)$$

where the constants C and D derive from the explicit form of the derivative of (13). Starting now from the linear combination of $x_o(\tau)$ and $y_o(\tau)$ given by

$$f_o(\tau) = Ax_o(\tau) + By_o(\tau) \quad (26)$$

the employ of the initial conditions leads us to

$$f_o(\tau) = x_o(-T/2)y_o(\tau) - y_o(-T/2)x_o(\tau) \quad (27)$$

From this expression we can extract the asymptotic behaviour of $f_o(\tau)$, i.e.

$$f_o(T/2) \sim \frac{D}{4\omega C} \exp(\omega T/2) \quad \text{if } T \rightarrow \infty \quad (28)$$

Now we need to take into account the lowest eigenvalue of the stability equation to obtain the value of the ratio of determinants. We can explain the situation as follows: the derivative of the topological solution does not quite satisfy the boundary conditions for the interval $(-T/2, T/2)$. When enforcing such a behaviour, the eigenstate is compressed and

the energy shifted slightly upwards. In such a case the zero-mode $x_o(\tau)$ is substituted for the $f_\lambda(\tau)$ which stands for

$$-\frac{d^2 f_\lambda(\tau)}{d\tau^2} + V''[x_c(\tau)]f_\lambda(\tau) = \lambda f_\lambda(\tau) \quad (29)$$

whenever

$$f_\lambda(-T/2) = f_\lambda(T/2) = 0 \quad (30)$$

Resorting now to the lowest order in perturbation theory we get

$$f_\lambda(\tau) \sim f_o(\tau) + \lambda \left. \frac{df_\lambda}{d\lambda} \right|_{\lambda=0} \quad (31)$$

so that

$$f_\lambda(\tau) = f_o(\tau) + \lambda \int_{-T/2}^{\tau} [x_o(\tau)y_o(s) - y_o(\tau)x_o(s)] f_o(s) ds \quad (32)$$

The asymptotic behaviour of $f_o(\tau)$, $x_o(\tau)$ and $y_o(\tau)$, together with the condition $f_\lambda(T/2) = 0$ allows us to find this lowest eigenvalue λ , i.e.

$$\lambda = 2\omega D^2 \exp(-\omega T) \quad (33)$$

The Gelfand-Yaglom method provides us with the final expression of the quotient of determinants whenever we choose for the frequency ν of the harmonic oscillator of reference the average of the frequencies of the central and lateral wells. In other words $\nu = 3\omega/2$. We can compare this situation with the well-grounded double-well model where the two minima of the potential are equivalent so that the aforementioned average is out of order. With this information we can write the one-instanton amplitude between the points $x_i = 0$ and $x_f = 1$, i.e.

$$\begin{aligned} &< x_f = 1 | \exp(-HT) | x_i = 0 > = \\ &\left(\frac{3\omega}{2\pi}\right)^{1/2} (2 \sinh 3\omega T/2)^{-1/2} \sqrt{S_{eo}} \sqrt{\frac{4}{3\pi}} \exp(-S_{eo}) \omega d\tau_c \end{aligned} \quad (34)$$

As expected we get a transition amplitude just depending on the point τ_c at which the instanton makes precisely the jump. Next we should take into account the configurations constructed out of instantons and antiinstantons which mimic the behaviour of a trajectory strictly derived from the euclidean equation of motion. In doing so we get an additional bonus since the integration over the centers of the string of instantons and antiinstantons is carried out in a systematic way. But this point is the subject of the next subsection.

B. The dilute-gas approximation.

Since the above calculations were carried out over a single instanton, it remains to identify the contributions associated with a string of widely separated instantons and anti-instantons along the τ axis. It is customarily assumed that these combinations of topological solutions represent no strong deviations of the trajectories just derived from the euclidean equation of motion without any kind of approximation. We evaluate the functional integral by summing over all such configurations, with k instantons and anti-instantons centered at points τ_1, \dots, τ_k whenever

$$-\frac{T}{2} < \tau_1 < \dots < \tau_k < \frac{T}{2} \quad (35)$$

If the regions where the instantons (anti-instantons) make the jump are narrow enough, the action of the proposed path is almost extremal. In this approach the total action is given by the sum of the k individual actions. This scheme is well-known in the literature where it appears with the name of dilute-gas approximation [7]. The translational degrees of freedom of the separated k topological configurations yield an integral of the form

$$\int_{-T/2}^{T/2} \omega d\tau_k \int_{-T/2}^{\tau_k} \omega d\tau_{k-1} \dots \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^k}{k!} \quad (36)$$

As regards the quadratic fluctuations we have now that the single ratio of determinants transforms into

$$\left(\frac{3\omega}{2\pi}\right)^{1/2} (2 \sinh 3\omega T/2)^{-1/2} \left\{ \frac{Det' [-(d^2/d\tau^2) + V''[x_c(\tau)]]}{Det [-(d^2/d\tau^2) + 9\omega^2/4]} \right\}^{-1/2} \longrightarrow$$

$$\left(\frac{3\omega}{2\pi}\right)^{1/2} \exp(-3\omega T/4) \left[\left\{ \frac{\text{Det}'[-(d^2/d\tau^2) + V''[x_c(\tau)]]}{\text{Det}[-(d^2/d\tau^2) + 9\omega^2/4]} \right\}^{-1/2} \right]^k \quad (37)$$

according to the limit of the factor associated with the harmonic oscillator when T is large. When going to the dilute-gas approximation one considers a set of instantons and anti-instantons so that each topological configuration starts where its predecessor ends. For our amplitude the total number k of instantons plus anti-instantons must be odd. The combinatorial factor F associated with the number of possible configurations corresponds to $F = 2^{(k-1)/2}$ because the closed paths starting and coming back to the point $x_o = 0$ require in a systematic way instanton-anti-instanton pairs. We remind the difference with the double-well potential where the instantons strictly alternate with the anti-instantons since the problem has only two minima and the combinatorial factor is not necessary. Now we can express the complete transition amplitudes for the triple-well potential so that

$$\langle x_f = 1 | \exp(-HT) | x_i = 0 \rangle = \left(\frac{3\omega}{4\pi}\right)^{1/2} \exp(-3\omega T/4) \sum_{j=0}^{\infty} \frac{(\omega T d)^{2j+1}}{(2j+1)!} \quad (38)$$

where as usual d stands for the instanton density, i.e.

$$d = \sqrt{\frac{8}{3\pi}} \sqrt{S_{eo}} \exp(-S_{eo}) \quad (39)$$

To sum up

$$\langle x_f = 1 | \exp(-HT) | x_i = 0 \rangle = \left(\frac{3\omega}{4\pi}\right)^{1/2} \exp(-3\omega T/4) \sinh(\omega T d) \quad (40)$$

If we return to the euclidean transition amplitude written in (1), the insertion of the pure discrete spectrum of energy eigenfunctions, namely

$$H|n\rangle = E_n|n\rangle \quad (41)$$

allows us to write that

$$\langle x_f = 1 | \exp(-HT) | x_i = 0 \rangle = \sum_n \exp(-E_n T) \langle x_f = 1 | n \rangle \langle n | x_i = 0 \rangle \quad (42)$$

Next we denote by E_o , E_1 and E_2 the energies of the ground-state and the two first excited levels of our problem. As the triple-well potential we are dealing with is even we

know that $\langle 1|x_i = 0 \rangle$ vanishes so that the limit $T \rightarrow \infty$ in (40) provides us with the following energy eigenvalues

$$E_o = \frac{3\omega}{4} - \omega d, \quad E_1 = \frac{3\omega}{4}, \quad E_2 = \frac{3\omega}{4} + \omega d \quad (43)$$

The tunneling transfers along the whole real axis the gaussian wave functions constructed over the points $x_- = -1$, $x_+ = 1$ and $x_o = 0$ so that the splitting term results proportional to the barrier-penetration factor. In addition the average of the harmonic frequencies over the non-equivalent minima of the potential serves as the central position for the splitting.

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